USING BLACK-BOX EXPERIMENTS TO DISCOVER A FORMULA FOR A FUNCTION[1]

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With the widespread use of graphic calculators and mathematics software packages, obtaining a rough graph of a function would seemingly be a snap for students. Yet, this is not the case! For example, many of our calculus students have accidentally done the following when using a graphing calculator or graphing software:

- Entered the function $f(x) = \dfrac{x^2 + x}{x - 1}$ as $f(x) = x^2 + x / x - 1$,
- Plotted the function as entered, and
- Accepted the graph as correct.

This error seems to occur because many of our students have a great difficulty in

- Recognizing differences between two algebraic expressions,
- Relating *obvious* visual properties of a function's graph to algebraic properties of a formula for the function, and
- Questioning results generated by a calculator or a computer.

We also find that many of our students have difficulty in both truly seeing and articulating differences between the graphs of various types of functions. For example,

- Visually, how are the graphs of power functions $p(x) = x^n$, n an integer, different? What about the graphs of the *even-degree* power functions?
- How are graphs of algebraic functions different from graphs of polynomial and rational functions?
- How are graphs of transcendental functions different from graphs of algebraic functions?

As a remedy for this situation, we present below a collection of exercises designed to develop and enhance students ability to predict algebraic properties of a function from visual properties of a function's graph. In many cases, this will enable the student to discover a formula for the function. To do this we will use the *hidden function* feature of the software package *TEMATH* 1.5.2. Using this feature, students can plot, evaluate, and compose a function without having access to an algebraic expression for the function. Such a function can be viewed as a black-box and algebraic properties of the function can be determined indirectly through graphical experimentation.

Graphical Experimentation 101

An unusual feature of a black-box function's graph, might lead a student to make a conjecture about an algebraic property of a formula for the function. Often a student's conjecture can be tested by a *graphical* experiment. A graphical experiment might consist of constructing a new function from a black-box function and then plotting the new function.

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[1]This article appeared in "Proceedings of the Eighth Annual International Conference on Technology in Collegiate Mathematics", Addison-Wesley Publishing Co., 1997, p. 186-190.

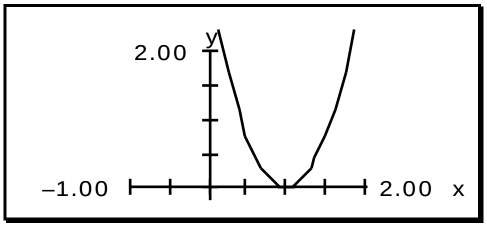


Figure 1 $y = f(x)$

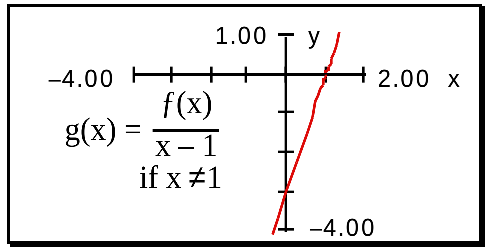


Figure 2 $y = g(x)$

For example, let $f(x)$ be the function whose graph is shown in Figure 1 above. Then $f(x)$ has a root at $x = 1$ and we might suspect that $f(x) = (x-1)g(x)$ for some simpler function $g(x)$. To test this idea, we can solve for $g(x)$ and plot the result. Suppose we do this and the graph of $g(x) = f(x)/(x-1)$ (if $x \neq 1$) is the line shown in Figure 2 above. By careful observation and measurement, we see that the line's y-intercept is –3 and its slope is 3; hence, $g(x) = 3x - 3$ (if $x \neq 1$). Suppose that by further experimentation we determine that $\lim_{x \to 1} g(x) = 0$. Then we can conclude that $g(x) = 3x - 3$ for all x. Hence, $f(x) = (x-1)g(x) = (x-1)3(x-1) = 3(x-1)^2$ is a formula for $f(x)$.

♦

Thus, information about an *algebraic* representation for a function can be deduced from simple graphical experiments. Listed below are the major features of a function's graph that can be used to infer properties of an *algebraic* representation for the function:

- x and y intercepts
- Domain
- Asymptotes
- Far-left and far-right behavior
- Local maximums and minimums, and vertical tangents

A student can verify conjectures and gain additional information by graphing the following related functions:

- $H(x) = (f \circ u)(x)$ — Transform the domain of $f(x)$ by a function $y = u(x)$.
- $K(x) = (v \circ f)(x)$ — Transform the range of $f(x)$ by a function $y = v(x)$.
- $Q(x) = \dfrac{f(x)}{x-a}$ if $x \neq a$ — Divide by a linear factor corresponding to a root.
- $P(x) = (x-a)f(x)$ — Multiply by a linear factor corresponding to a vertical asymptote.
- $Delta(x) = f(x+1) - f(x)$ — Construct the unit difference function.

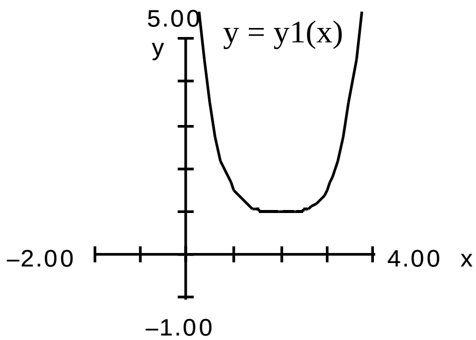
Note that if $u(x) = c\,x + h$, then $H(x)$ translates and scales $f(x)$ horizontally. Similarly, if $v(x) = d\,x + k$, then $K(x)$ translates and scales $f(x)$ vertically.

Finally, a student could use a data-analysis/modeling approach to determine a formula for the function. For example, a student could try to fit a polynomial, rational, exponential, or logarithmic model to a table of (x, y)-values sampled from the function's graph.

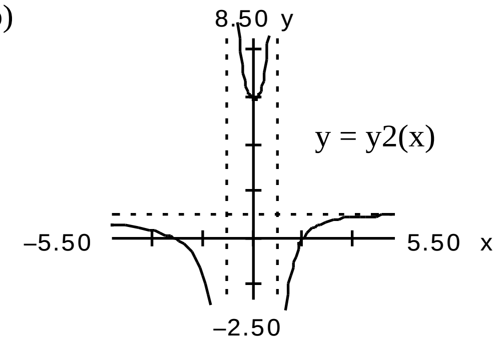
Sample Exercises

Set 1 Use graphing of related functions to find a formula for the given function.

a)



b)



<u>Solutions</u>:

a) • Shift $y_1(x)$ vertically and horizontally to the origin (see Figure 3 below).
   • Scale the translated function so that it passes through (1,1) as shown in Figure 4. Call this function $f(x)$.

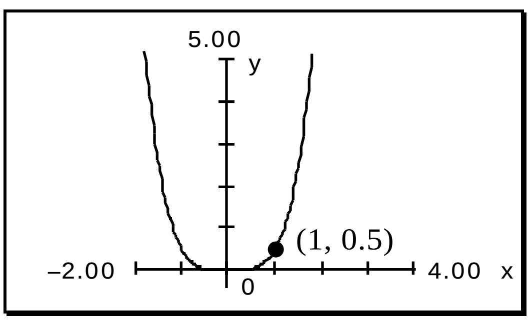


Figure 3 $y = y_1(x+2) - 1$

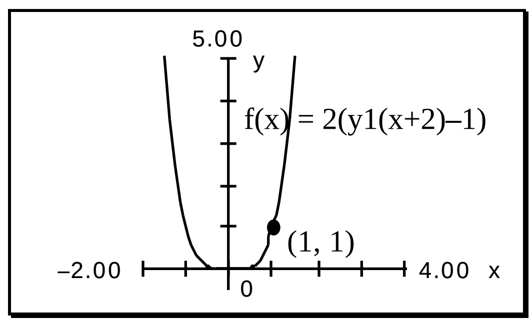


Figure 4 $y = f(x)$

   • The symmetry and simplicity of $f(x)$'s graph implies that $f(x)$ is an *even* degree power function.
   • However, the graph of $Delta(x) = f(x+1) - f(x)$, shown in Figure 5 below, is not linear; hence, $f(x)$ is not a quadratic function.

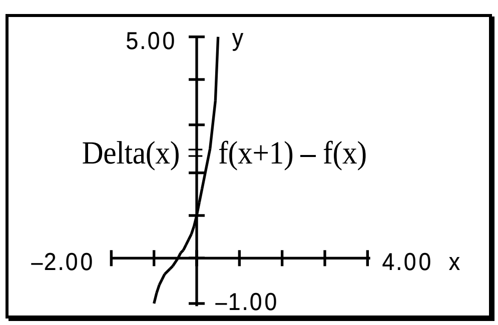


Figure 5 $y = Delta(x)$

   • Confirm that $f(x)$ is quartic by comparing its graph and the graph of $y = x^4$.

☞ $f(x) = 2(y_1(x+2)-1) = x^4 \quad y_1(x) = \frac{1}{2}(x-2)^4 + 1.$

b) • Remove $y_2(x)$'s vertical asymptotes by multiplying by $(x-1)(x+1)$ if $x \neq \pm 1$.
• Remove its roots by dividing by $(x-2)(x+3)$ if $x \neq 2, -3$ (see Figure 6 below).

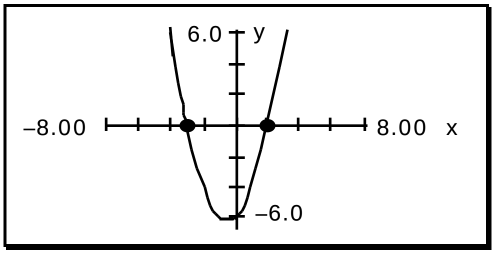


Figure 6 $y = (x-1)(x+1)y_2(x)$

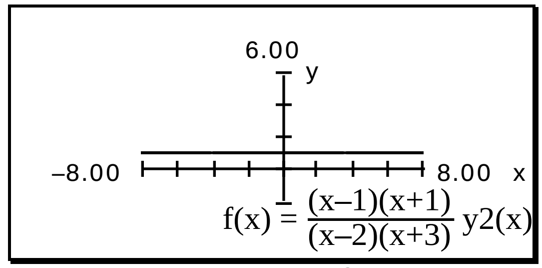


Figure 7 $y = f(x)$

• Notice that the graph of $f(x) = \frac{(x-1)(x+1)}{(x-2)(x+3)} y_2(x)$, shown in Figure 7 above, is the line $y = 1$ if $x \neq \pm 1, 2, -3$.

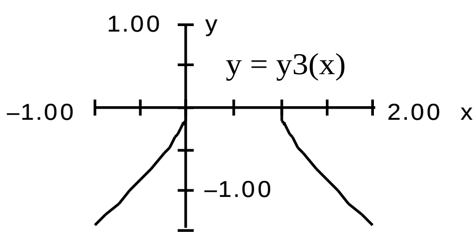
☞ $y_2(x) = \frac{(x-2)(x+3)}{(x-1)(x+1)}$.

♦

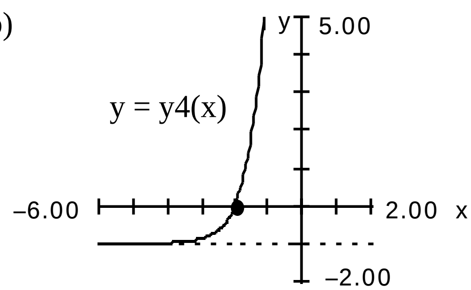
If the domain of a function has large gaps, and its graph has vertical tangents, it is likely that it has an algebraic representation that involves the use of a radical. If the function's graph is very steep and has a horizontal asymptote, it is likely that the function is exponential.

Set 2 Find formulas for the following algebraic and exponential functions.

a)



b)



Solutions:

a) • Since $y_3(x)$ is non-positive and has vertical tangents at 0 and 1, plot $y = y_3(x)^2$.
• Divide the resulting function by $x(x-1)$ to remove its roots (see Figures 8 and 9).
• Notice that the graph of $f(x) = \frac{y_3(x)^2}{x(x-1)}$ is the horizontal line $y = 1$ if x $[0,1]$.

☞ $y_3(x) = -\sqrt{x^2 - x}$.

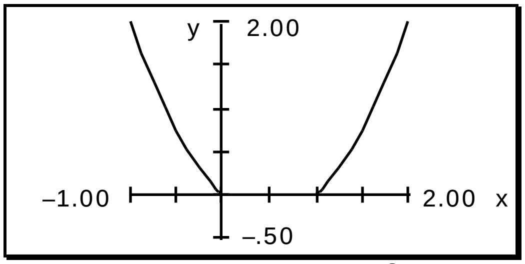


Figure 8 $y = y_3(x)^2$

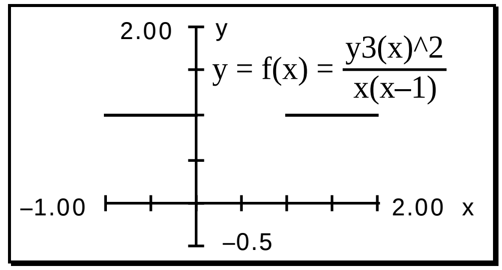


Figure 9 $y = f(x)$

b) • Shift $y_4(x)$ upward by 1 so that its graph is asymptotic to the x-axis.

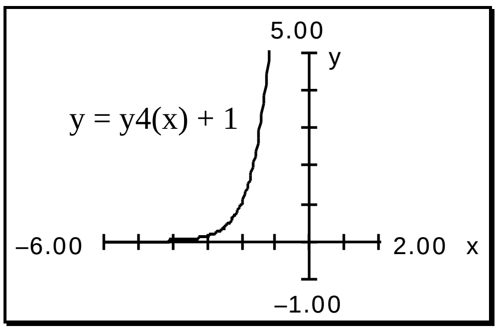


Figure 10 $y = y_4(x) + 1$

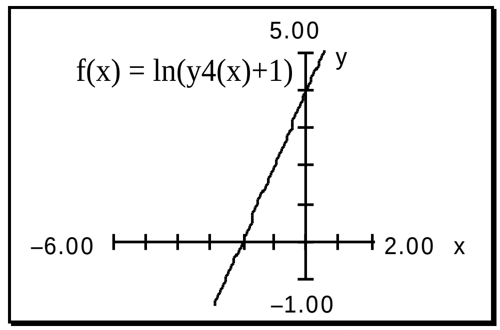


Figure 11 $y = f(x)$

- Notice that the translated graph shown in Figure 10 looks exponential but is steeper than the graph of $y = e^x$. To verify this conjecture, plot $f(x) = \ln(y_4(x) + 1)$.
- Notice that $f(x)'s$ graph shown in Figure 11 is the line $y = 2x + 4$.

☞ $y_4(x) = e^{2x+4} - 1$.

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As a project, students could be asked to describe a sequence of graphical experiments for determining if a function belongs to a given family of functions. For example, a function $f(x)$ belongs to the family $l(x) = \ln(ax + b)$, $a \neq 0$, if the graph of $y = e^{f(x)}$ is a ray.

Set 3 Describe a sequence of graphical experiments that can be used to determine whether or not $f(x)$ belongs to the given family of functions.

a) $c(x) = \sqrt[3]{ax^2 + bx + c}$, $a \neq 0$

b) $q(x) = \dfrac{a}{(x-h)^2 + d^2} + k$, $ad \neq 0$

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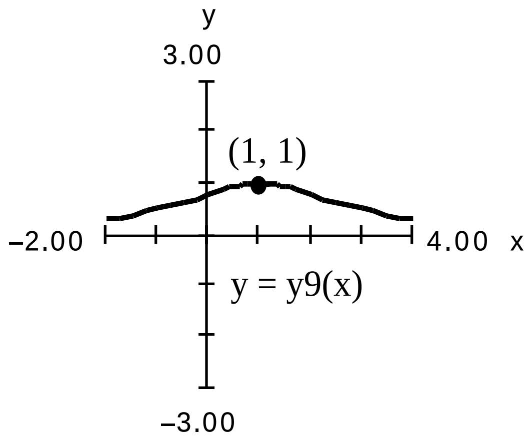
Bibliography

[1] *TEMATH - Tools for Exploring Mathematics Version 1.5* by Robert Kowalczyk and Adam Hausknecht, Brooks/Cole, 1993.

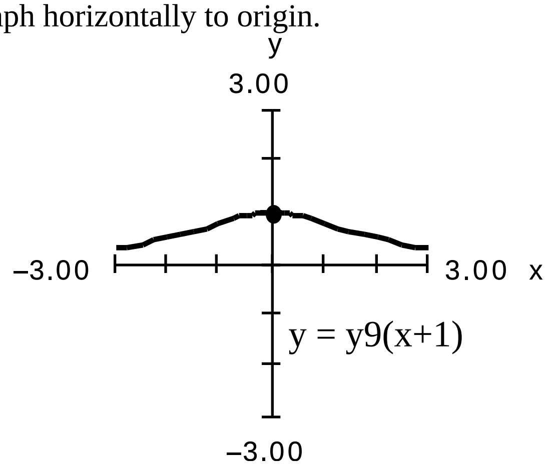
[2] *A Guide to TEMATH - Tools for Exploring Mathematics* by Robert Kowalczyk and Adam Hausknecht, Brooks/Cole, 1991.

[3] *Mathematical Explorations with TEMATH, by* Robert Kowalczyk and Adam Hausknecht, to be published by Brooks/Cole in 1996.



Solution:

- Shift the function's graph horizontally to origin.



- Observe that the graph has $y = 0$ as a horizontal asymptote; hence, the function is *not* polynomial. Moreover, it rises and falls too slowly to be exponential or logarithmic. Thus, assume it's rational. Since it has no roots or vertical asymptotes it must be a rational function of the form $y = \dfrac{a}{(bx)^2 + c^2}$.
- Try fitting various rational functions of this form to $f(x) = y_9(x+1)$. Hint: Since $f(0) = 1$, $a = c^2$ and the values of b and c are determined by two other points on f's graph.

☞ $f(x) = \dfrac{4}{x^2 + 4}$ $\quad y_9(x)$.

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Major Features of a Function's Graph
<=>
Algebraic Properties of a Formula for the Function

- *x* intercept & *y* intercepts
- Domain
- Asymptotes
- Vertical Tangents
- Far Left and Far Right Behavior
- Local Maximums and Minimums

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A student could also use a data-analysis/modeling approach to determine a formula for the function. For example, a student could

- Generate a table of (x, y)-values for the function,
- Use interpolation to find a polynomial passing through the (x, y)-values, and
- Verify the result by comparing the complete graph of the interpolating polynomial with the complete graph of $f(x)$.

A student could also try to fit the table of (x, y)-values using other models including rational and exponential models.

[1]This article appeared in "Proceedings of the Eighth Annual International Conference on Technology in Collegiate Mathematics", Addison-Wesley Publishing Co., 1997, p. 186-190.

Brief Vitae

Adam O. Hausknecht, who co-authored TEMATH with Prof. Kowalczyk, received his Ph. D. in algebra from U. C. Berkeley in 1975. In Fall of 1982, he joined the Mathematics Department of the University of Massachusetts -Dartmouth where he helped form the Department of Computer and Information Sciences. He has taught a wide range of mathematics and computer science courses including college algebra, calculus, differential equations, numerical mathematics, combinatorics, abstract algebra, Pascal, C, data structures, compiler design, theory of computation, and computer graphics. His interests include developing mathematics software for education, computer algebra systems, and noncommutative algebra. He enjoys bicycling, hiking, mysteries, baking, and listening to the blues (but not all at the same time).

Robert Kowalczyk, who co-authored TEMATH with Prof. Hausknecht, received a B.A. in mathematics from UMASS-Dartmouth and a Ph.D. in applied mathematics from Brown University. He worked at Raytheon Company before joining the faculty at the University of Massachusetts-Dartmouth in 1975. He teaches courses in calculus, statistics, and numerical analysis and he uses the computer as a demonstration tool and a as a visualization tool in all his classes. His interests include developing educational software for learning mathematics. He is an avid traveler, photographer, square dancer, and round dancer.